COMP 761: Lecture 8 – Combinatorics

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Problem

You have m identical balls and a row of n labeled buckets. How many ways can you put the balls in the buckets? (Buckets can be empty.)

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Vincent has also prepared an optional list of practice problems where we can give feedback on your solutions, but they will not count towards a grade

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- They get big fast...

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- After that, there are $n - 2$ places for x_3, etc.
- Multiplying all these options, the total number of ways to assign the things is:

$$n(n-1)(n-2)\cdots(2)(1) = n!\,.$$

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- So overall, we have:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

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$$\binom{10}{0} = \frac{10!}{0!10!} = 1, \quad \binom{10}{10} = \frac{10!}{10!0!} = 1, \quad \binom{10}{1} = \frac{10!}{1!9!} = 10.$$

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- 10! is huge, so let's avoid it.
- We know that $10! = 10 \cdot 9 \cdot 8 \cdot (7!)$, so we have

$$\frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8}{3!} = \frac{10 \cdot 9 \cdot 8}{6} = 5 \cdot 3 \cdot 8 = 120.$$

n choose $n-k$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

n choose $n - k$

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- From this formula, we see that:

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- That should give the same result, so $\binom{n}{n-k} = \binom{n}{k}$.

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- Now just 2 people left, so they are automatically Group C.
- Overall number of ways:

$$\binom{6}{2}\binom{4}{2} = \frac{6!}{2!4!}\frac{4!}{2!2!} = \frac{6!}{2!2!2!} = \frac{720}{8} = 90.$$

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- In general, the number of way to divide n people into Group 1 of size n_1, Group 2 of size n_2, ..., up to Group k of size n_k:

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

(this is called a *multinomial* coefficient)

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- Overall we get:

$$\binom{n}{k} \cdot k! = \frac{n!}{k!(n-k)!} \cdot k! = \frac{n!}{(n-k)!}.$$

Pascal's Triangle

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

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							1							
						1		1						
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				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

- nth row (starting at 0), kth entry is $\binom{n}{k}$.

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							1							
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			1		4		6		4		1			
		1		5		10		10		5		1		
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1		7		21		35		35		21		7		1

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- Can see $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = n$.

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1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
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1	7	21	35	35	21	7	1

- nth row (starting at 0), kth entry is $\binom{n}{k}$.
- Can see $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = n$.
- Symmetric since $\binom{n}{n-k} = \binom{n}{k}$.

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$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

- nth row (starting at 0), kth entry is $\binom{n}{k}$.
- Can see $\binom{n}{0} = \binom{n}{n} = 1$ and $\binom{n}{1} = n$.
- Symmetric since $\binom{n}{n-k} = \binom{n}{k}$.
- Notice that each number is the sum of the two directly above it:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Pascal's Identity

Prove that:

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&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k \cdot (n-1)!}{k!(n-k)!} + \frac{(n-k) \cdot (n-1)!}{k!(n-k)!} \\
&= \frac{n \cdot (n-1)!}{k!(n-k)!}
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&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{k \cdot (n-1)!}{k!(n-k)!} + \frac{(n-k) \cdot (n-1)!}{k!(n-k)!} \\
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Combinatorial proof:

- Consider picking a group of k things from n things.
- Two cases: either pick the first thing or not.

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- We can also think of the denominator as

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- What about in general?
- We can do the same thing arranging $n-1$ '|'s and m 'O's:

$$\binom{m+n-1}{m}$$

Next time!

Graph Theory I