COMP 761: Lecture 14 – Calculus II

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October 5, 2020

Problem

Let X, A be $n \times n$ matrices, with X the matrix with all entries equal to t^2. What is the derivative $\frac{d}{dt}\operatorname{tr}(XA)$?

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

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- Office hours right after class today.

Double derivatives

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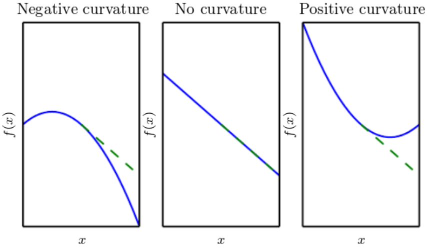
- Graphically, measures the *curvature*:



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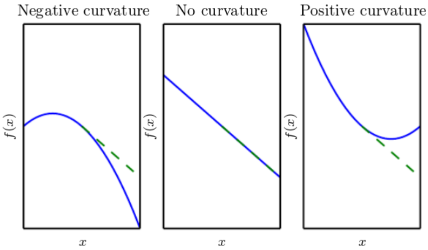

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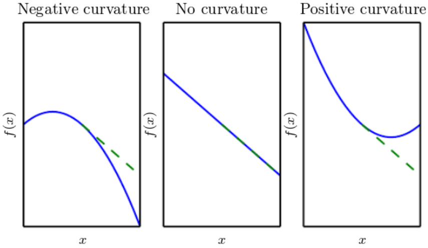

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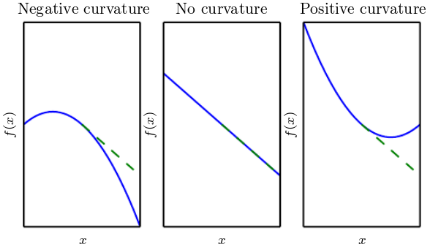

- If a function $f(x)$ has $f''(x) \geq 0$ for all x, we say it is *convex*.
- If a function $f(x)$ has $f''(x) \leq 0$ for all x, we say it is *concave*.
- **Most functions are neither convex nor concave.**

Jensen's inequality

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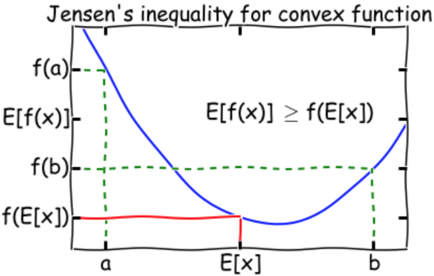
- *Jensen's inequality*: if $f(x)$ is a convex function, then

$$\frac{f(x_1) + \ldots + f(x_n)}{n} \geq f\left(\frac{x_1 + \ldots + x_n}{n}\right)$$



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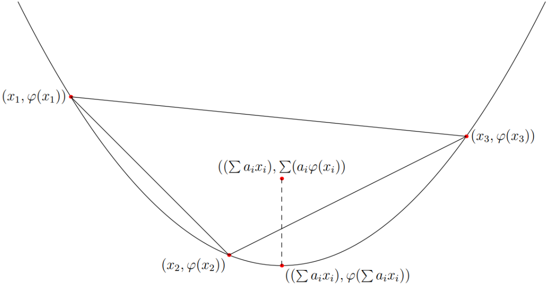

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- If $f(x)$ is concave, then $-f(x)$ is convex, so

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therefore

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- What if $f''(x) = 0$? (both convex and concave)
- Then, $f(x) = ax + b$ (linear function) and:

$$\frac{f(x_1) + \ldots + f(x_n)}{n} = f\left(\frac{x_1 + \ldots + x_n}{n}\right)$$

Problem

Prove the AM-GM Inequality:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n}, \quad \text{for } a_1, a_2, \ldots, a_n \geq 0$$

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- That means we could apply Jensen's inequality to $-\log(x)$.

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- Taking exponentials of both sides gives us AM-GM.

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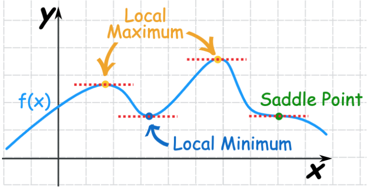

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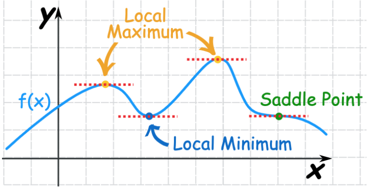

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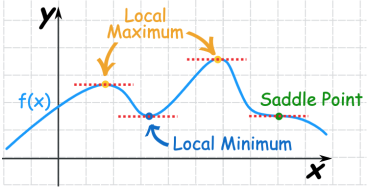

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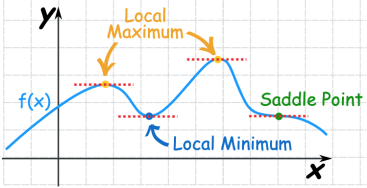

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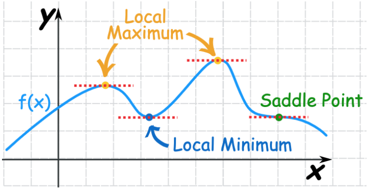

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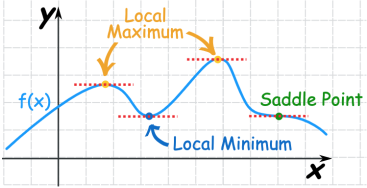

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- If $f'(x) = 0$ and $f''(x) < 0$, then local minimum.
- If $f'(x) = 0$ and $f''(x) = 0$, then neither (saddle point).

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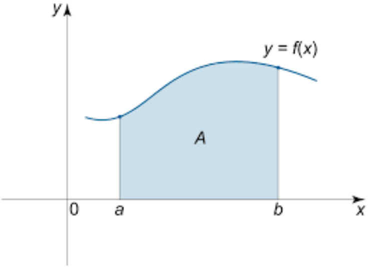
$$\int_a^b f(x)dx$$

is the area under the curve of the function $f(x)$ between $x = a$ and $x = b$.



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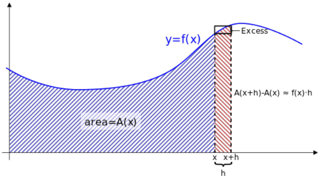

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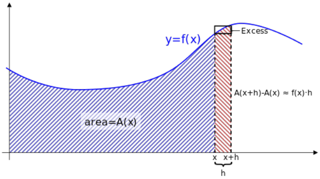

- Can see that for $h \approx 0$, we have $A(x+h) \approx A(x) + hf(x)$, so

$$\frac{d}{dx}A(x) = \lim_{h\to 0}\frac{hf(x)}{h} = f(x).$$

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- We have $\frac{d}{dt}\frac{1}{3}t^3 = t^2$, so

$$\int_{-1}^{1} t^2 dt = \frac{1}{3}(1)^3 - \frac{1}{3}(-1)^3 = \frac{2}{3}.$$

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- For example:

$$\frac{\partial}{\partial x}(x^2 + 2y^2 + xy) = 2x + y$$
$$\frac{\partial}{\partial y}(x^2 + 2y^2 + xy) = 4y + x.$$

The gradient

The gradient

- The *gradient* ∇f of a multivariable function $f(x_1, \dots, x_n)$ is the vector of partial derivatives with respect to the variables:

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- For example:

$$\nabla(x^2 + y^2 + 2z^2) = \left[\begin{array}{ccc} 2x & 2y & 4z \end{array} \right].$$

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so

$$\begin{aligned}
f(x+\epsilon_x, y+\epsilon_y) &\approx f(x+\epsilon_x, y) + \epsilon_y \frac{\partial}{\partial y} f(x+\epsilon_x, y) \\
&\approx \left(f(x,y) + \epsilon_x \frac{\partial}{\partial x} f(x,y) \right) + \epsilon_y \frac{\partial}{\partial y} f(x+\epsilon_x, y) \\
&\approx f(x,y) + \epsilon_x \frac{\partial}{\partial x} f(x,y) + \epsilon_y \frac{\partial}{\partial y} f(x,y) \\
&= f(x,y) + \nabla f \cdot [\epsilon_x, \epsilon_y].
\end{aligned}$$

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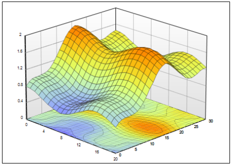
- In general:

$$f(x_1 + \epsilon_1, \ldots, x_n + \epsilon_n) \approx f(x_1, \ldots, x_n) + (\nabla f) \cdot \begin{bmatrix} \epsilon_1 & \cdots & \epsilon_n \end{bmatrix}.$$



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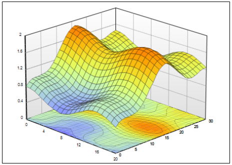

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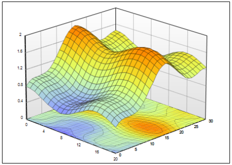

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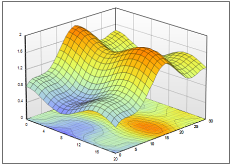

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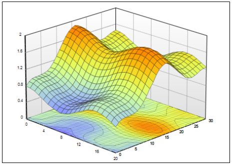

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- Likewise, greatest *decrease* when $\begin{bmatrix}\epsilon_1 & \cdots & \epsilon_n\end{bmatrix}$ pointing along negative gradient $(-\nabla f)$.

Multivariate chain rule

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$$\begin{aligned} \frac{d}{dt}f &= (\nabla f) \cdot \begin{bmatrix} \frac{d}{dt}x_1(t) & \cdots & \frac{d}{dt}x_n(t) \end{bmatrix} \\ &= \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}. \end{aligned}$$

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- Example: if $x = t^2$ and $y = e^t$, then

$$\frac{d}{dt}(x^2 + y^2) = (2x)(2t) + (2y)(e^t)$$
$$= (2t^2)(2t) + (2e^t)(e^t)$$
$$= 4t^3 + 2e^{2t}.$$

Next time!

Calculus III