COMP 761: Lecture 23 – Heaps

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October 28, 2020

Problem

What is the average-case time for Quicksort? Reminder, the method is:

- 1) Take an element x from somewhere in the list L.
- 2) Go through L one-by-one and *pivot* using x: that is, put anything lower than x in L_1, anything higher than x in L_2.
- 3) Sort L_1 and L_2 recursively.
- 4) Combine L_1, x, L_2, in that order.

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Problem set 4 posted, due Nov. 6

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- We have $X = \sum_{i<j} X_{ij}$, where X_{ij} is the indicator variable for whether z_i and z_j are compared (1 if they are, 0 if not).
- By linearity of expectation:

$$\mathbb{E}[X] = \sum_{i<j} \mathbb{E}[X_{ij}] = \sum_{i<j} p(\text{comparing } z_i \text{ and } z_j)$$

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- z_i and z_j are compared if and only if we pick z_i or z_j as a pivot *before* picking any of the other numbers between z_i and z_j.
- Therefore, we have:

$$\begin{aligned}
&p(\text{comparing } z_i \text{ and } z_j) \\
&= p(\text{picking } z_i \text{ first out of the nums between } z_i \text{ and } z_j) \\
&\quad + p(\text{picking } z_j \text{ first out of the nums between } z_i \text{ and } z_j) \\
&= \frac{1}{j-i+1} + \frac{1}{j-i+1} \\
&= \frac{2}{j-i+1}.
\end{aligned}$$

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- So we have:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{i<j} \mathbb{E}[X_{ij}] \\
&= \sum_{i<j} \frac{2}{j-i+1} \\
&= 2\sum_{i} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-i+1} \right) \\
&\leq 2n \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} \right) \\
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- Where we used $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = O(\log n)$ from last time.
- Since best-case running time is $\Theta(n \log n)$, the average running time is not just $O(n \log n)$, it is $\Theta(n \log n)$.

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- So each node has a *depth* = the number of edges between it and the root.

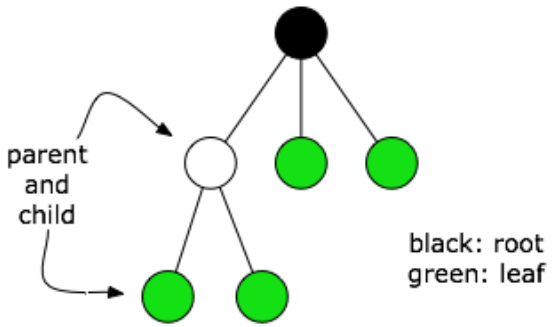

Figure: tree data structure

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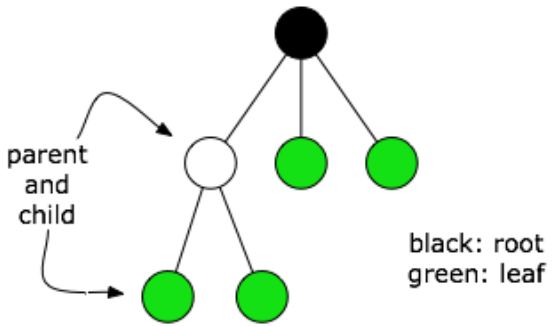

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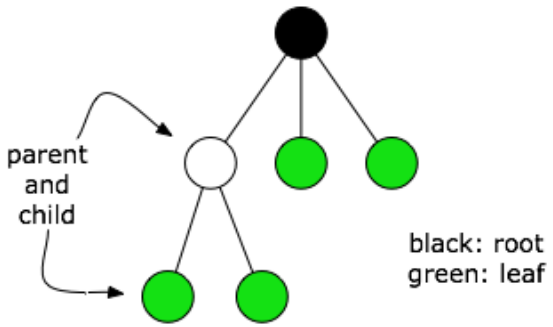

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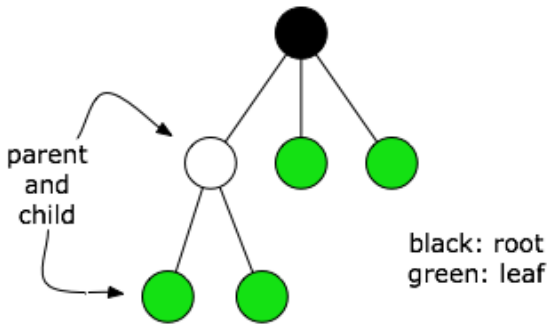

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- If w is the parent of v, then v is said to be a *child* of w.

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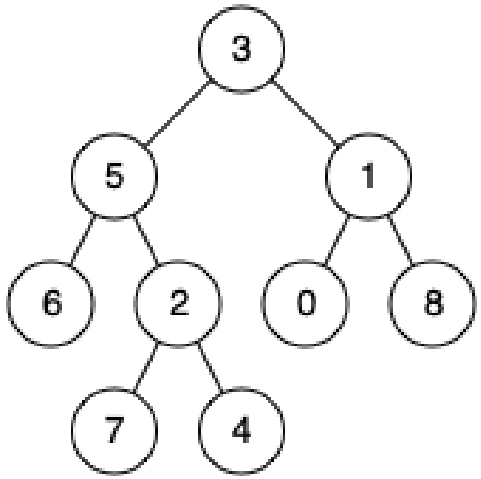
- Often nodes in a binary tree are used to store data.



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- The tree could be perfectly balanced with 2 nodes of depth 1, 4 of depth 2, etc.
- In this case, if depth is d, we have

$$\begin{aligned} n &= 1 + 2 + 4 + \ldots + 2^d \\ &= \frac{2^{d+1} - 1}{2 - 1} \\ &= 2^{d+1} - 1, \end{aligned}$$

so the minimum depth is $d = \Theta(\log n)$.

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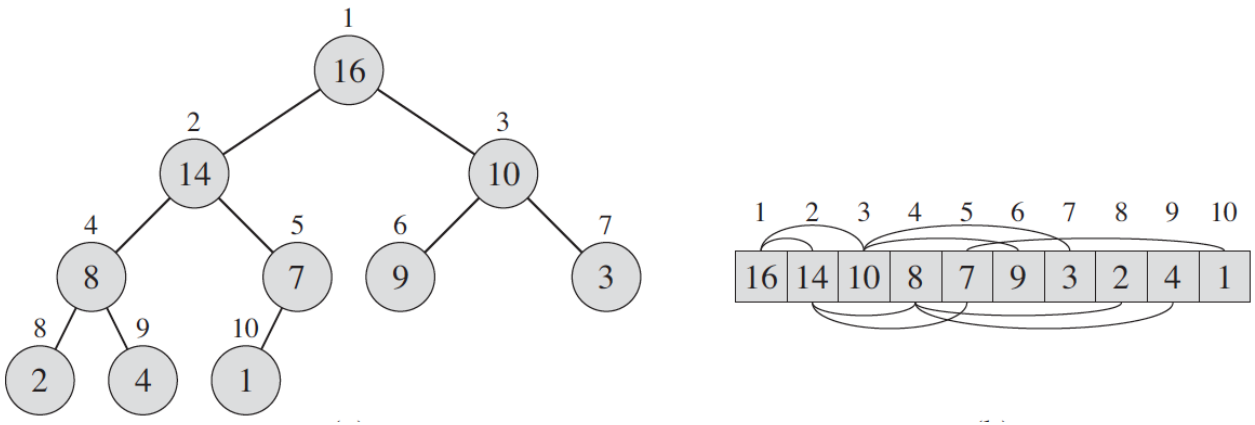
- The root node always must have the largest key.



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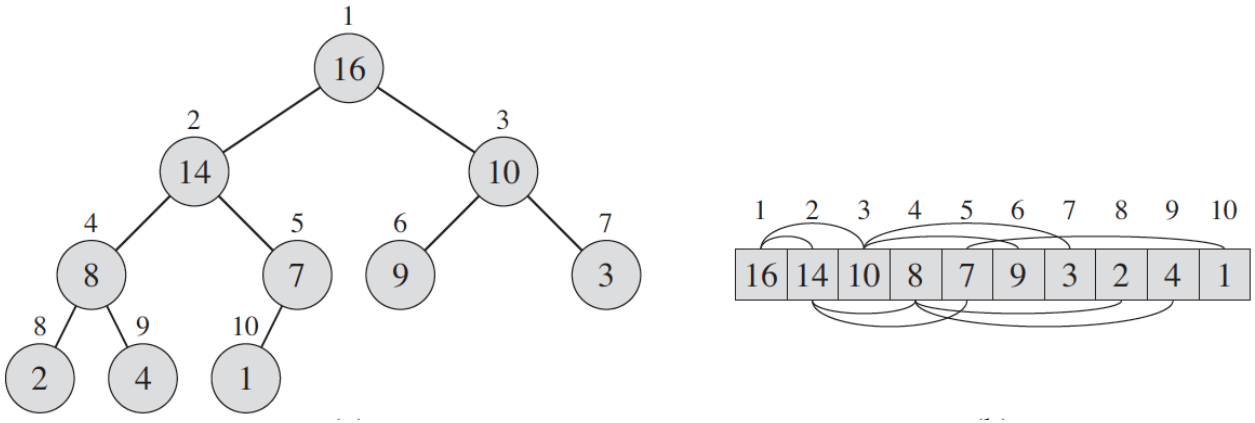

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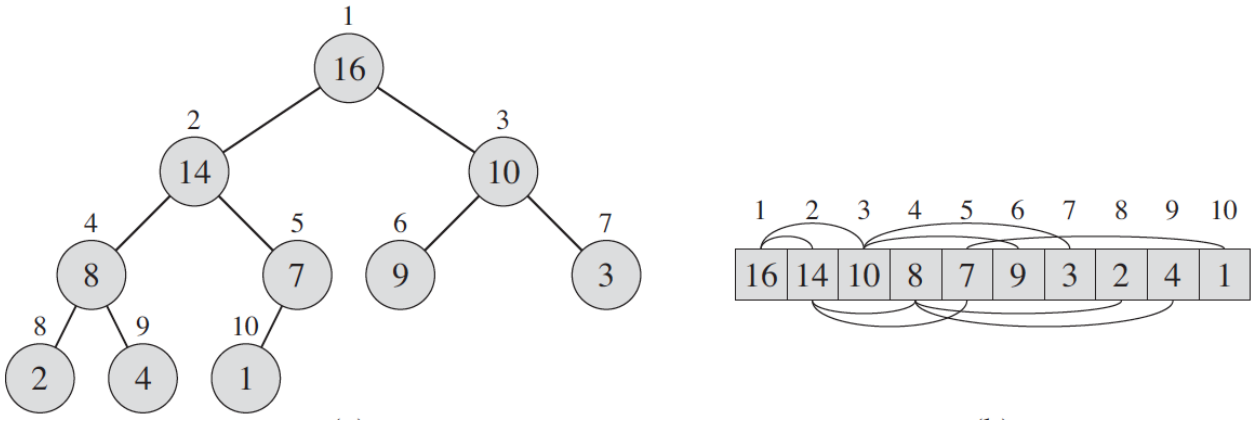

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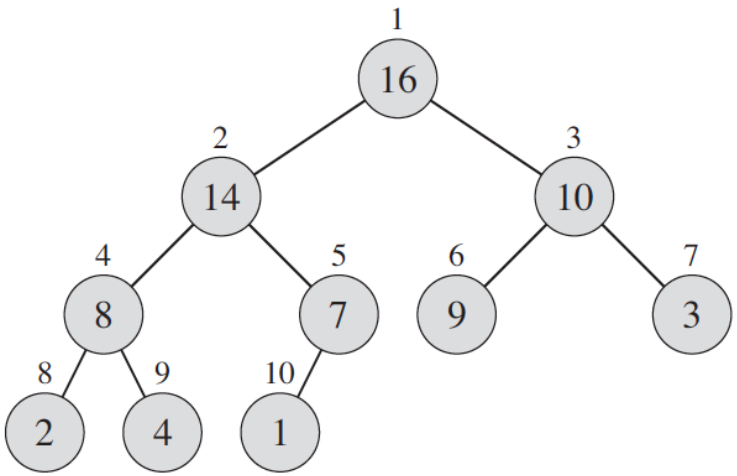

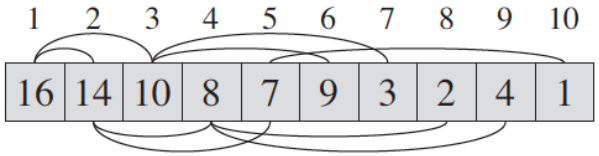

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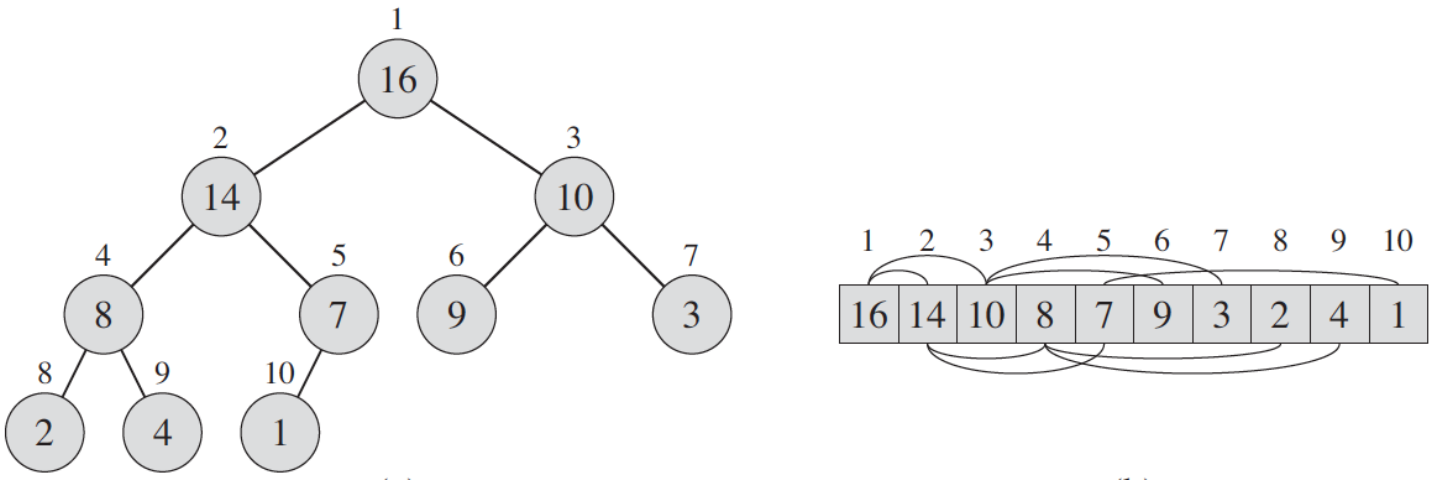

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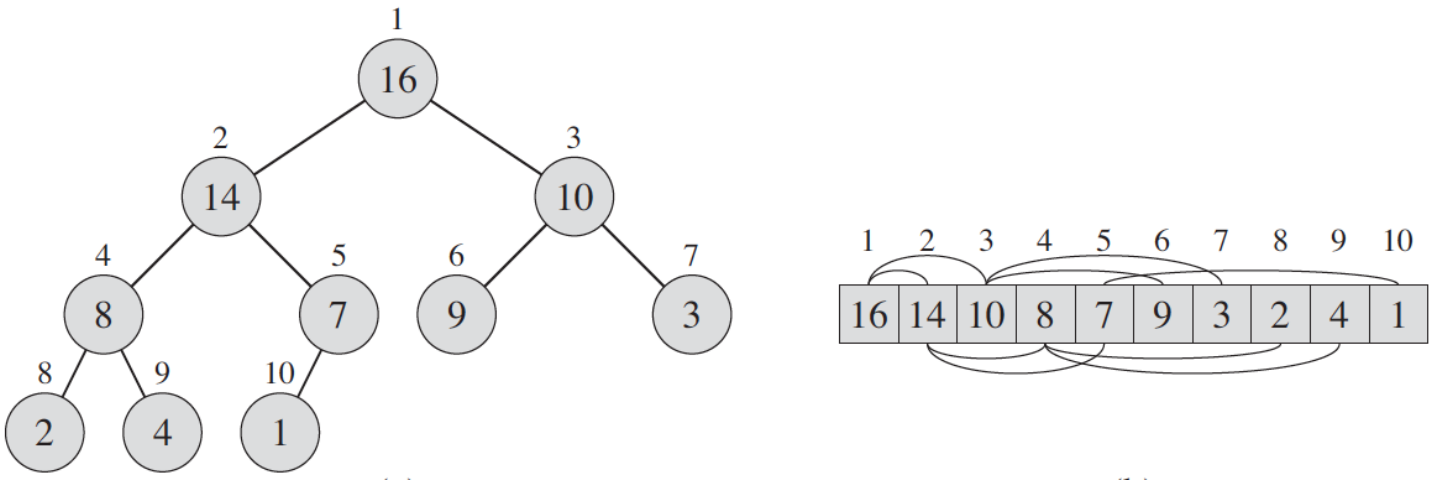

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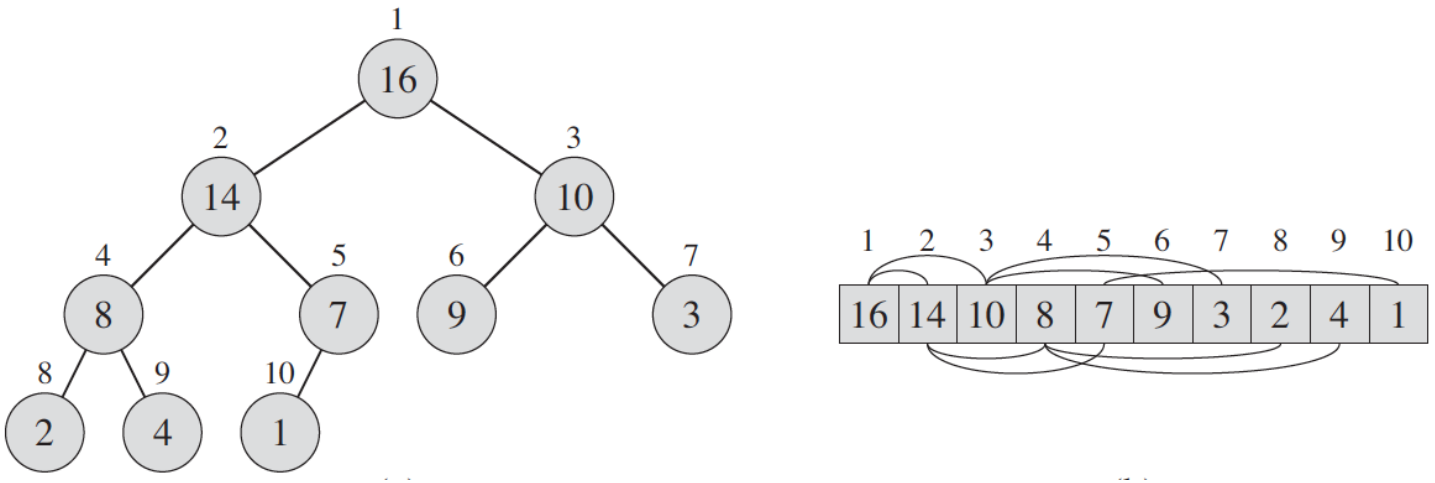

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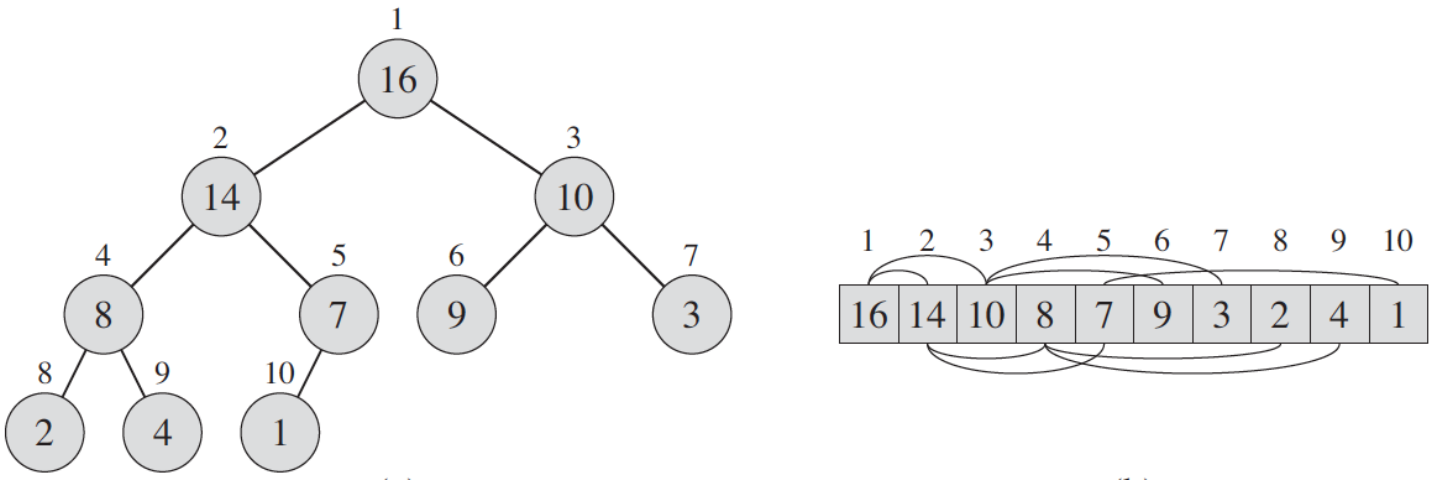

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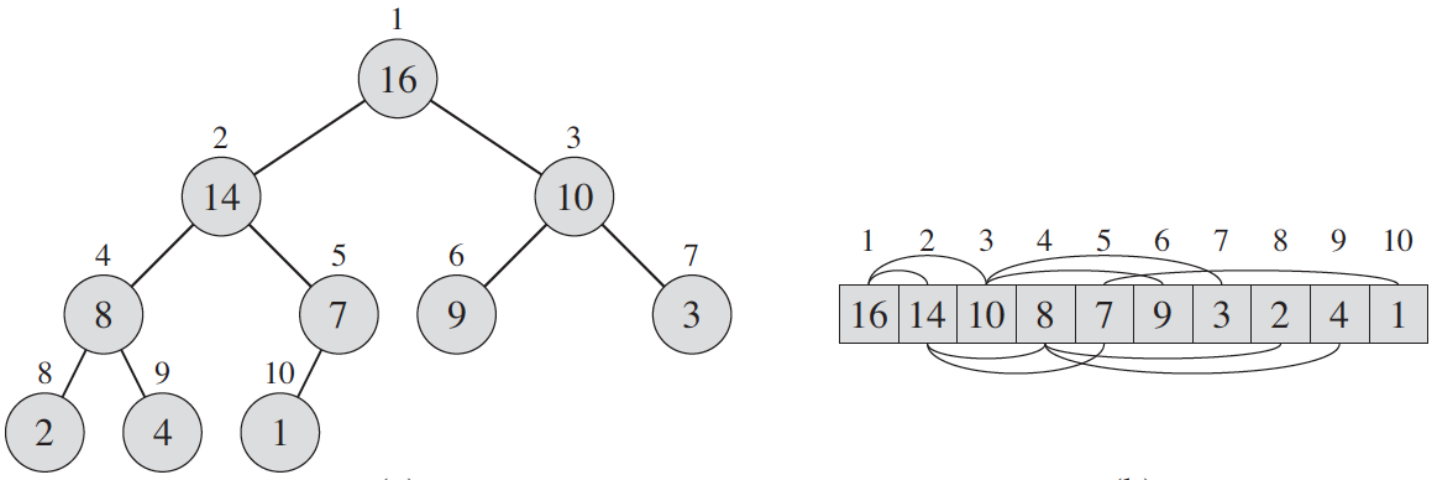

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- How to get from a node to its left child?
- Just double the index!
- For the right child, double the index and add 1.

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- Now for the next row up.
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- Similar process for rows higher up.

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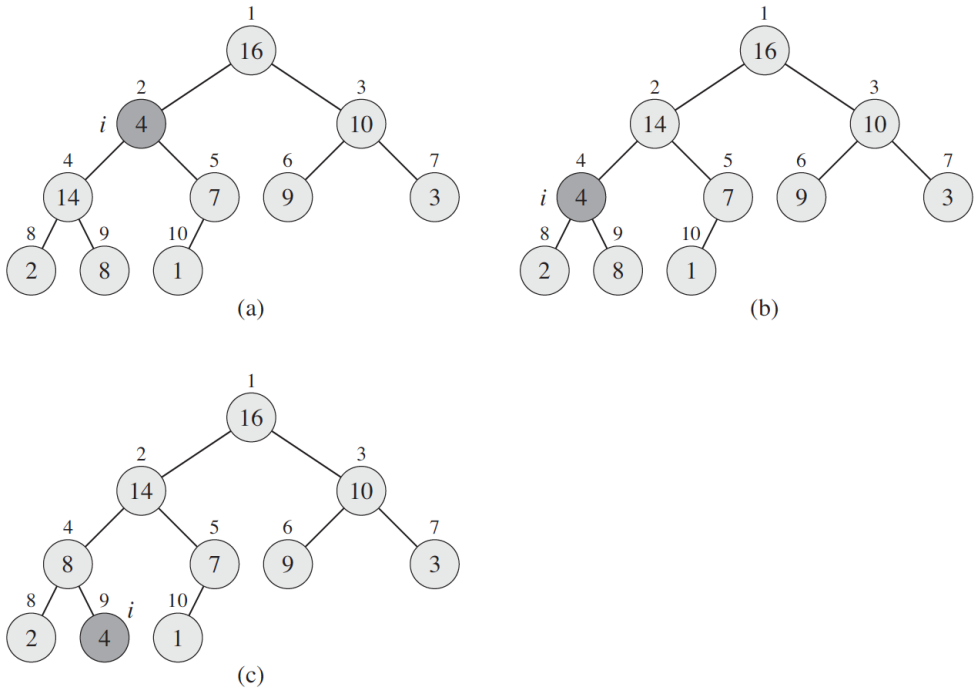
- Continue swapping until the max property is satisfied or the key reaches a leaf node.



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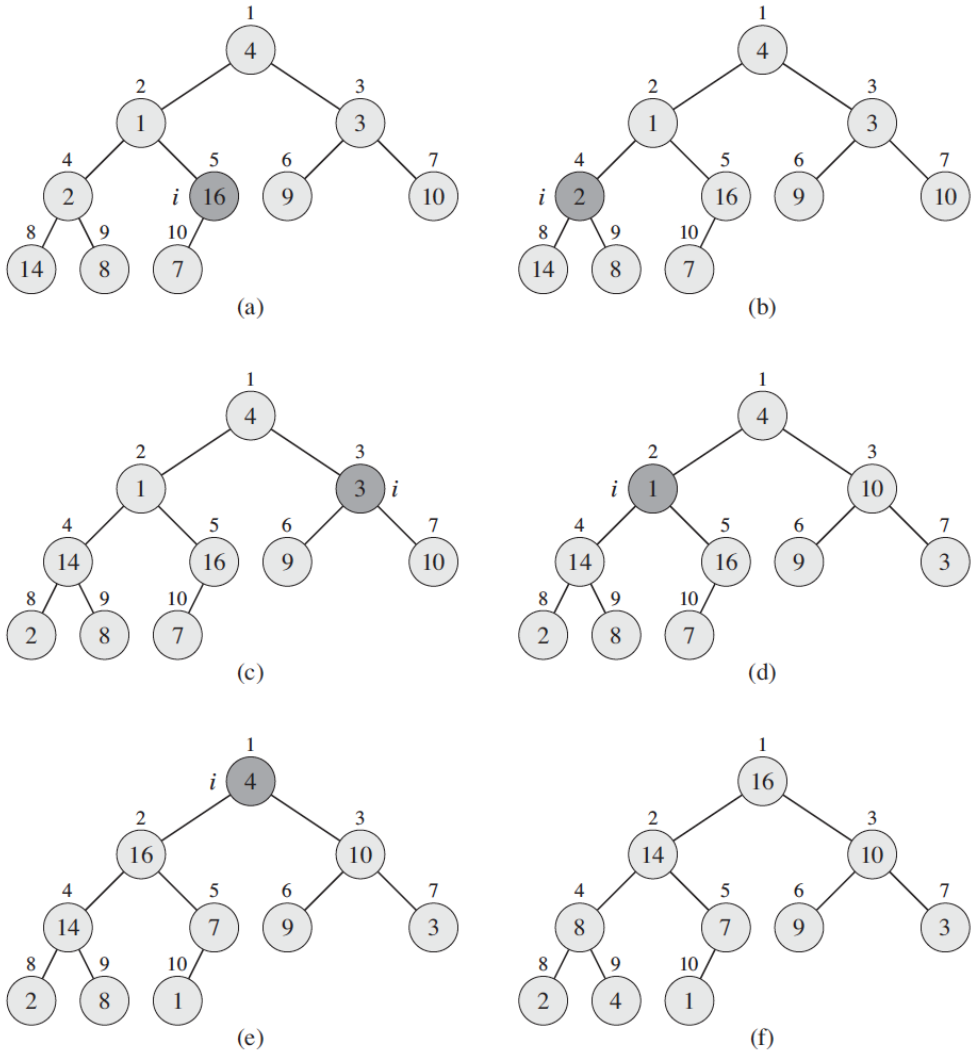
- We can do this to order the whole heap!



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- How many nodes are at depth k?

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- How long does it take?
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- So $O(n)$ time to build the heap.

Priority queues

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- A *max-priority queue* is a data structure S that allows you to perform the following operations
  - Insert(S, x), inserting key x into S.
  - Maximum(S), returns the maximum of S.
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- (Likewise, a min-heap can be used to implement a min-priority queue.)

Next time!

Graph algorithms I