COMP 761: Lecture 29 – Binary Search Trees II

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November 11, 2020

Problem

Suppose that we insert $\{1, 2, \ldots, n\}$ into a binary search tree in random order. What is the expected height?

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

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- Problem set 5 is out!

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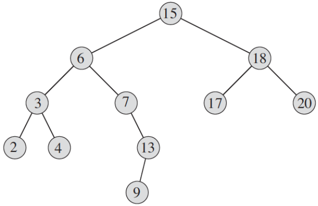
- Office hours: Vincent Thu at 10:30 am, David Fri at 10 am

Review: Binary search trees

- A *binary search tree* is a binary tree, each node storing a *key*.



- We require that for every node v:
  - The left subtree has all nodes less than or equal to v.
  - The right subtree has all nodes greater than or equal to v.

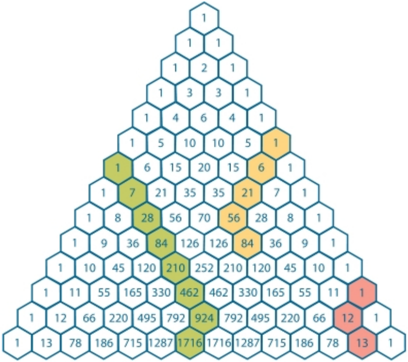
Review: Expected height

- We have a lot of algorithms running in $O(h)$.
- Maximum height with n keys: $h = n - 1$.
- Minimum height: $h = O(\log n)$.
- Let's consider a *typical* binary search tree.
- Suppose that we insert $\{1, 2, \ldots, n\}$ into a binary search tree in random order. What is the expected height?

Review: Hockey stick identity

- Hockey stick identity in our proof:

$$\sum_{i=0}^{n-1} \binom{i+k}{k} = \binom{n+k}{k+1}.$$



Review: Jensen's inequality

- We will also use another form of Jensen's inequality - if f is convex, then:

$$\mathbb{E}[f(x)] \geq f(\mathbb{E}[x]).$$

- This is essentially the same as the weighted form of Jensen's inequality we have already seen:

$$\sum_{i=1}^{n} p_i f(x_i) \geq f\left(\sum_{i=1}^{n} p_i x_i\right)$$

if p_i are nonnegative with $\sum_{i=1}^{n} p_i = 1$.

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- What is the root of the tree?

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- How can we use this?

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$$X_n = 1 + \max(X_{i-1}, X_{n-i}).$$

Note that i is itself a random variable.

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$$= \frac{2}{n}\sum_{i=0}^{n-1}(\mathbb{E}[Y_i] + \mathbb{E}[Y_{n-1-i}]) = \frac{4}{n}\sum_{i=0}^{n-1}\mathbb{E}[Y_i].$$

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- What is the right-hand side equal to?

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- Is $f(x) = 2^x$ convex/concave/neither?
- We have $2^x = (e^{\log 2})^x = e^{(\log 2)x}$, so

$$\frac{d}{dx}2^x = \frac{d}{dx}e^{(\log 2)x} = (\log 2)e^{(\log 2)x}$$
$$\frac{d^2}{dx^2}2^x = (\log 2)\frac{d}{dx}e^{(\log 2)x} = (\log 2)^2 e^{(\log 2)x} = (\log 2)^2 2^x > 0.$$

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$$\frac{1}{4}\binom{n+3}{3} \geq \mathbb{E}[Y_n] = \mathbb{E}[2^{X_n}] \geq 2^{\mathbb{E}[X_n]}.$$

- Therefore $\mathbb{E}[X_n] = O\left(\log\left(\frac{1}{4}\binom{n+3}{3}\right)\right) = \boxed{O(\log n)}$, since $\log(p(n)) = O(\log n)$ for any polynomial $p(n)$ (e.g. $\log(n^3) = 3\log n$).

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- However, this kind of average-case analysis doesn't necessarily help with any particular tree.
- We will now see a way to *make sure* that $h = O(\log n)$ not $O(n)$.

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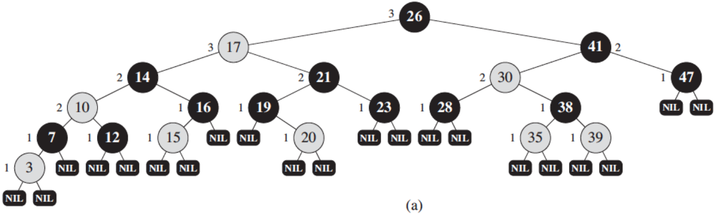

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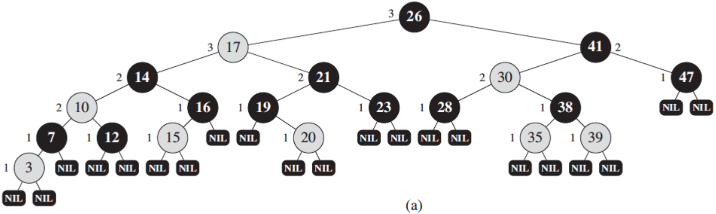

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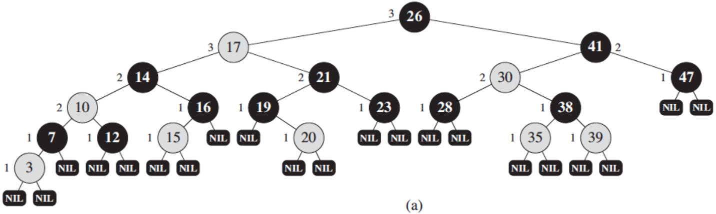

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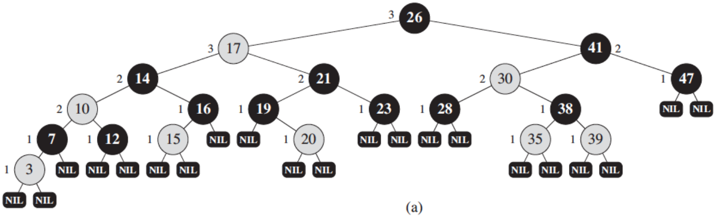

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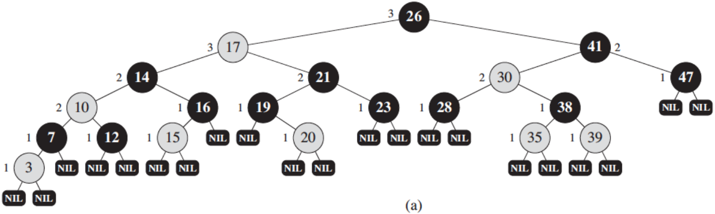

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  - The root is black.

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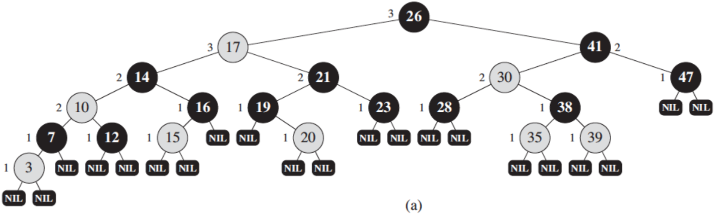

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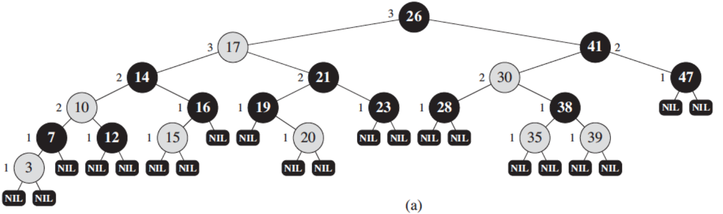

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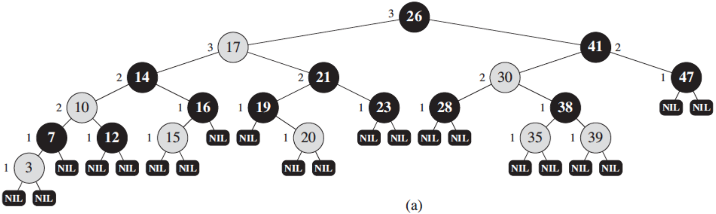

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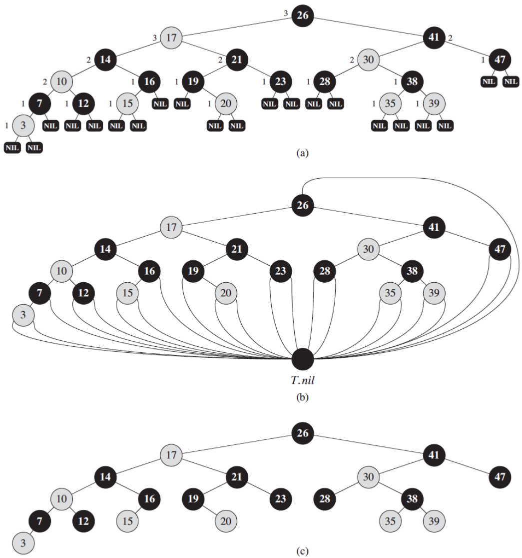

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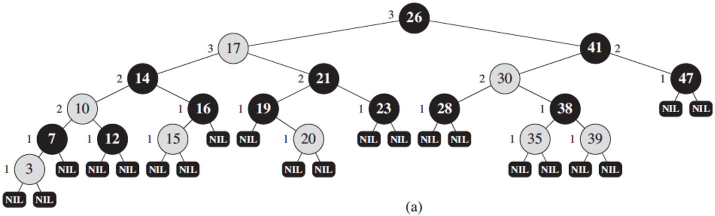

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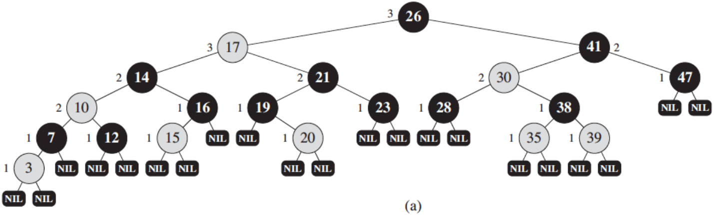

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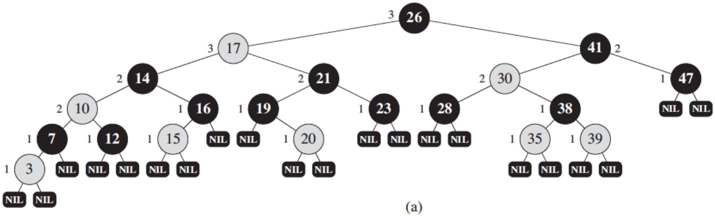

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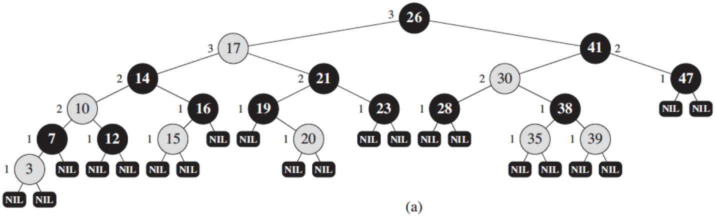

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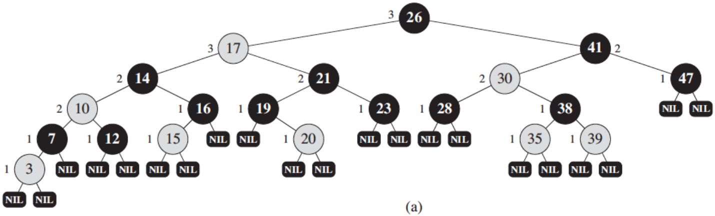

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- We say that an *internal node* of a red-black tree is any node that isn't a leaf (so any node containing a key).

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- Since height of y and z less than x, inductive hypothesis implies the subtrees rooted at y and z each have at least $2^{\text{bh}(x)-1} - 1$ internal nodes.
- So subtree rooted at x has at least

$$1 + \left(2^{\text{bh}(x)-1} - 1\right) + \left(2^{\text{bh}(x)-1} - 1\right) = 2^{\text{bh}(x)} - 1$$

  internal nodes, finishing the induction.

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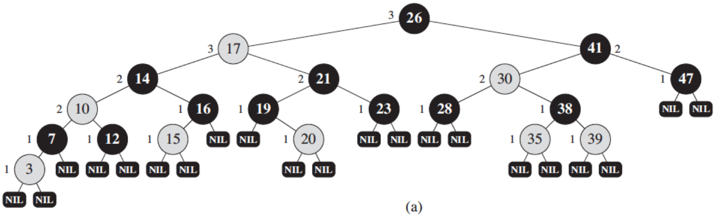
- Remember we assumed that in a red-black tree, if a node is red, both its children are colored black.
- How does this help?
- We know that $\text{bh}(x)$ is at least half the height of x, so

$$n + 1 \geq 2^{\text{height}(x)/2},$$

implying

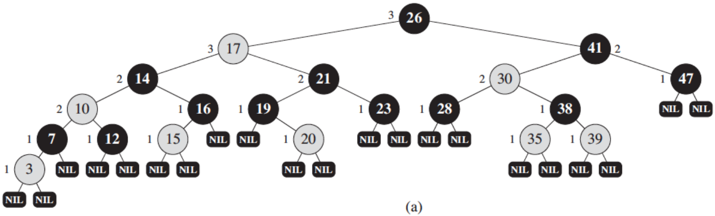
$$\text{height}(x) \leq 2 \log_2(n + 1) = O(\log n).$$

Tree operations



(a)

Tree operations



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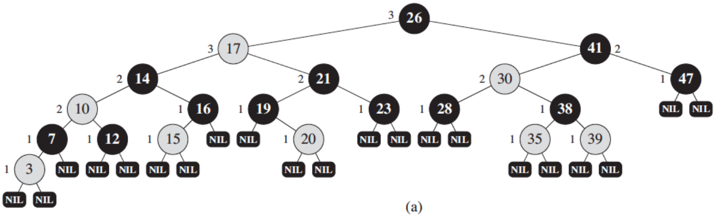

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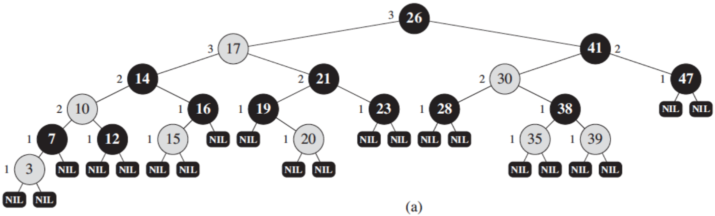

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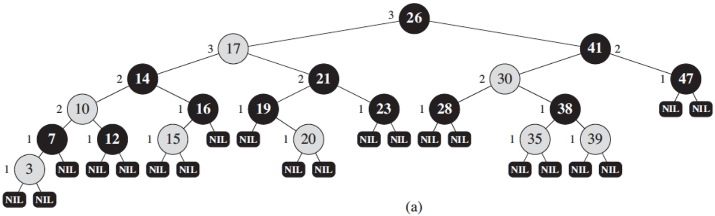

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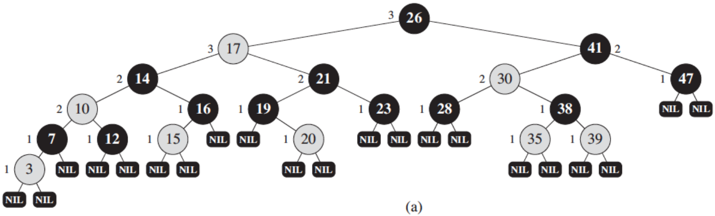

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- So all these operations run naturally in time $O(\log n)$.
- Insert and Delete must be changed so the red/black conditions work.

Rotations

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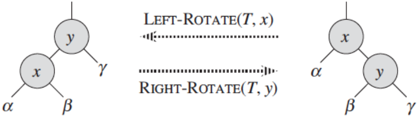
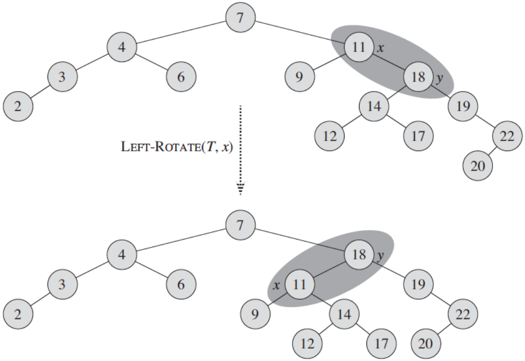
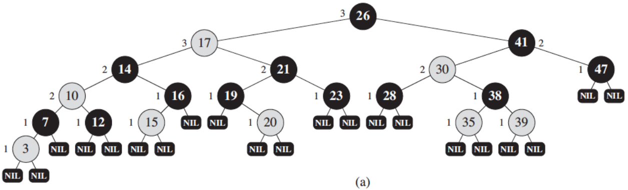
- We will use the following operations, called *left rotation* and *right rotation*:





Insert



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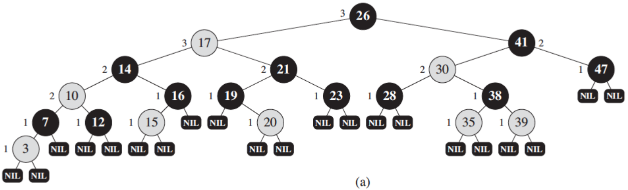

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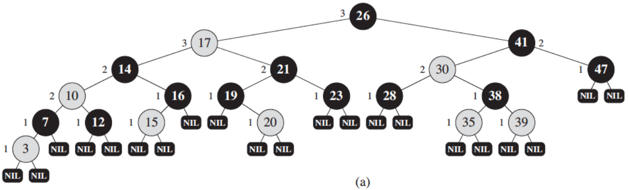

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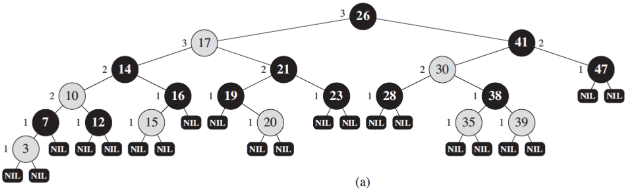

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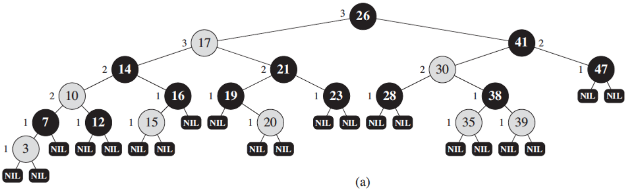

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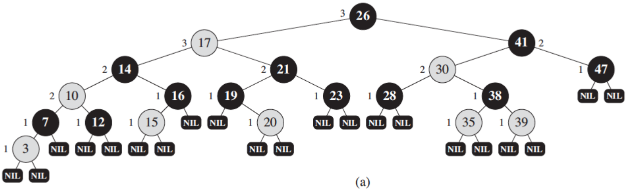

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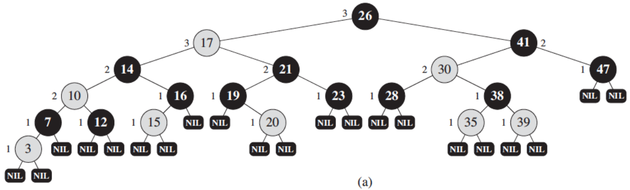

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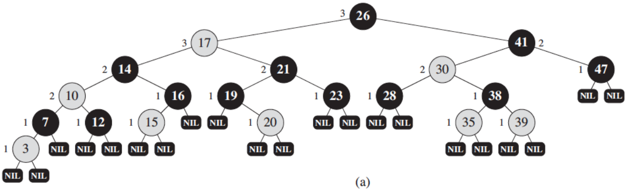

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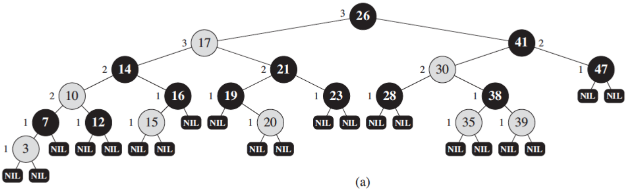

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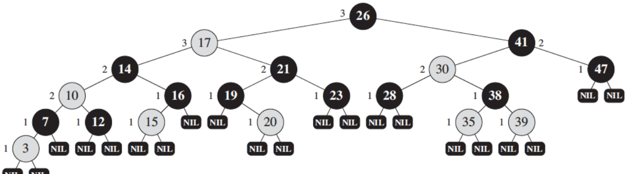

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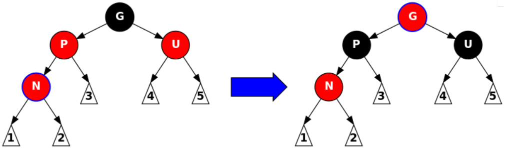
- **Case 1.** The uncle of the new node is red.
- **Case 2.** The uncle is colored black, and the new node is a right child.
- **Case 3.** The uncle is colored black, and the new node is a left child.

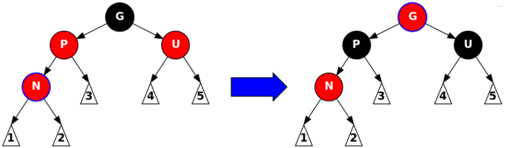
Insert

Case 1. The uncle of the new node is red.



Insert

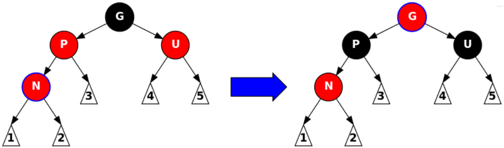
Case 1. The uncle of the new node is red.



- We swap colors as shown.

Insert

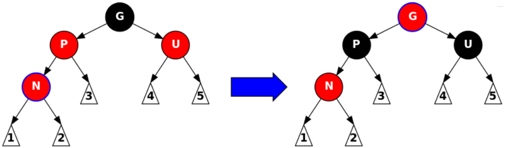
Case 1. The uncle of the new node is red.



- We swap colors as shown.
- Can check doesn't lead to any new violations.

Insert

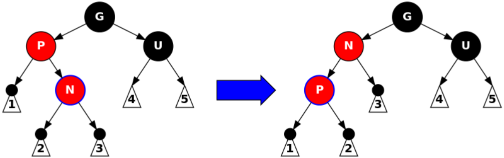
Case 1. The uncle of the new node is red.



- We swap colors as shown.
- Can check doesn't lead to any new violations.
- Except that the grandparent may now be a red violation if its own parent is red – in that case, we can recursively repeat the correction process we are now doing.

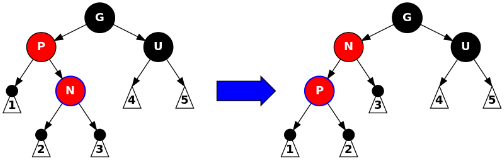
Insert

Case 2. The uncle is colored black, and the new node is a right child.



Insert

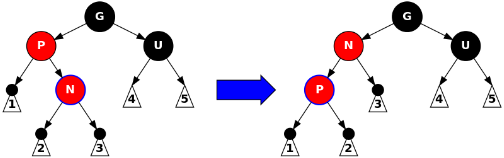
Case 2. The uncle is colored black, and the new node is a right child.



- We run a left rotation on the parent to reduce to the next case, case 3.

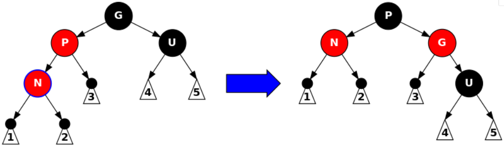
Insert

Case 2. The uncle is colored black, and the new node is a right child.



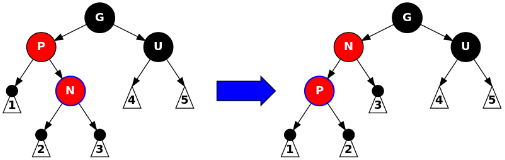
- We run a left rotation on the parent to reduce to the next case, case 3.

Case 3. The uncle is colored black, and the new node is a left child.



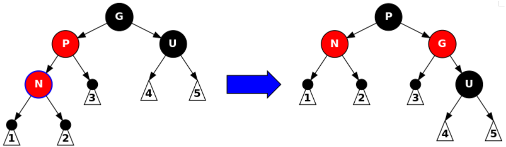
Insert

Case 2. The uncle is colored black, and the new node is a right child.



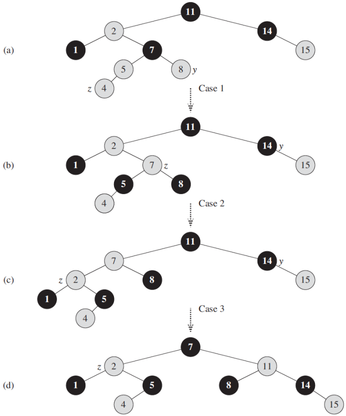
- We run a left rotation on the parent to reduce to the next case, case 3.

Case 3. The uncle is colored black, and the new node is a left child.



- We run a right rotation on the grandparent, and then swap the colors of the parent and grandparent.

Insert summary



Red-black conditions:

- The root is black.
- Both children of a red node are colored black.
- For each node, all paths from the node to the descendant leaves have the same number of black nodes.

Delete

Delete

- Delete in a red-black tree is a bit more complicated.

Delete

- Delete in a red-black tree is a bit more complicated.
- But there is a way to do it in $O(\log n)$ time.

Delete

- Delete in a red-black tree is a bit more complicated.
- But there is a way to do it in $O(\log n)$ time.
- So all our operations on a red-black tree run in time $O(\log n)$.

Next time!

Hashing