COMP 761: Lecture 33 – Computational Complexity

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November 20, 2020

Problem

Is there an efficient algorithm that determines whether an undirected, unweighted graph has a cycle that includes every vertex exactly once?

(Please don't post your ideas in the chat just yet, we'll discuss the problem soon in class.)

Course Announcements

Decision problems

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- The *computational complexity* of a problem refers to how hard it is to solve that problem.
- Proving that something can be solved fast generally means building an algorithm that solves it exactly that fast.
- Proving that something *cannot* be solved fast is sometimes harder.

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- Note that, for example, $\Theta(n \log n)$ algorithms are polynomial time because they are $O(n^2)$.
- We write the set of decision problems (yes/no problems) that can be solved in polynomial time as P.

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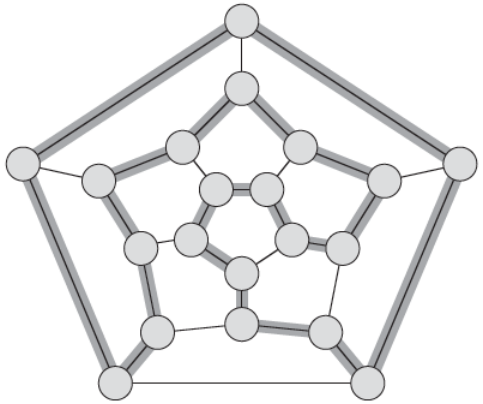
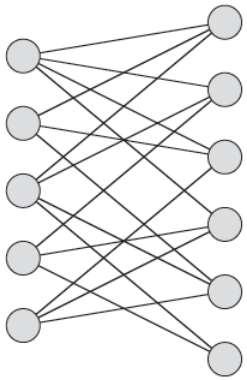


(a)



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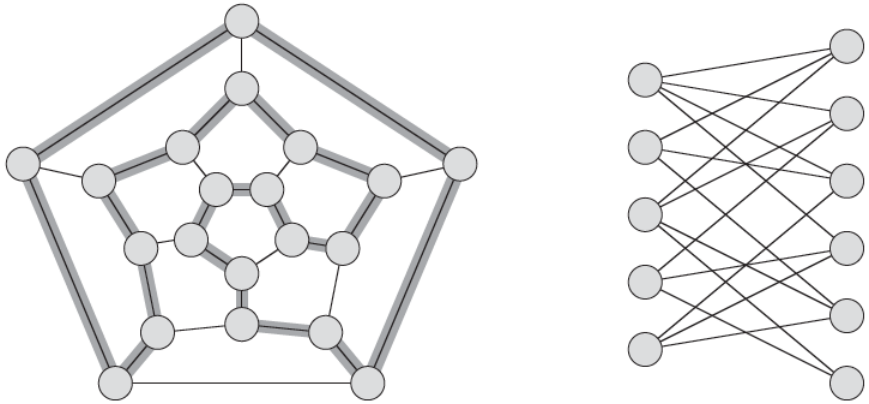

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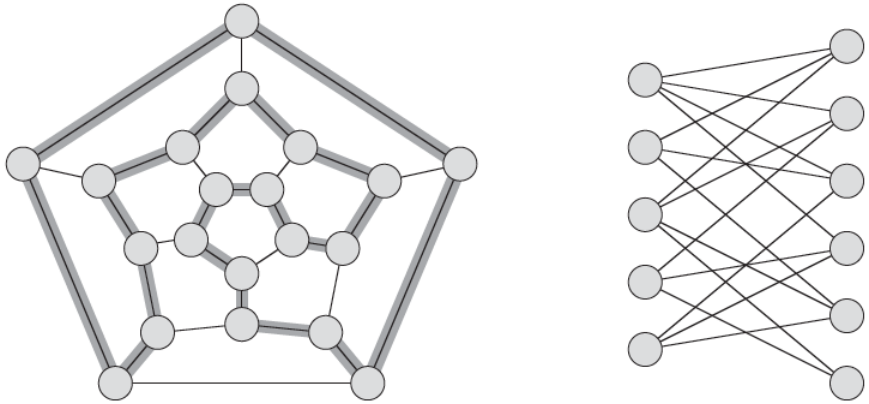

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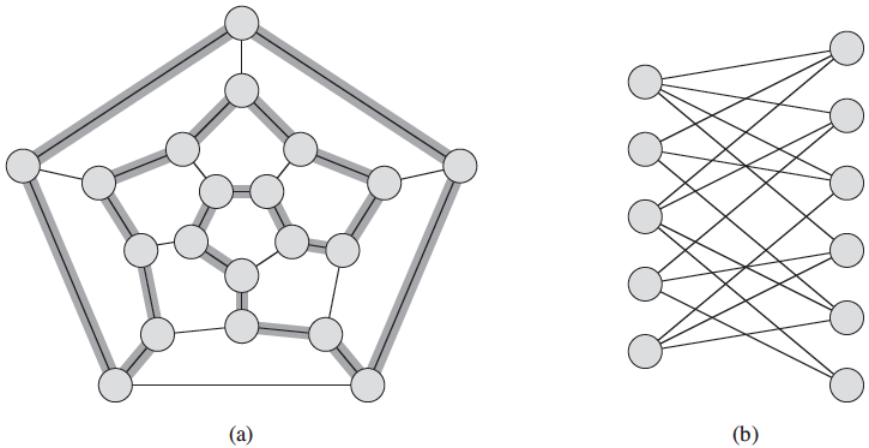

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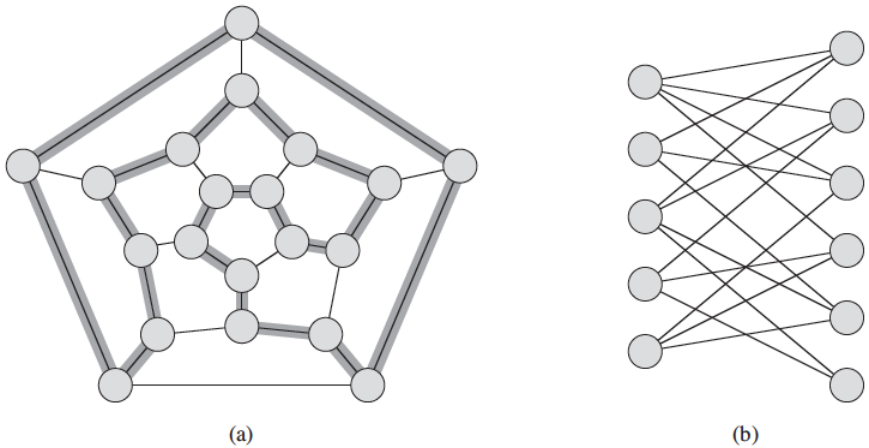

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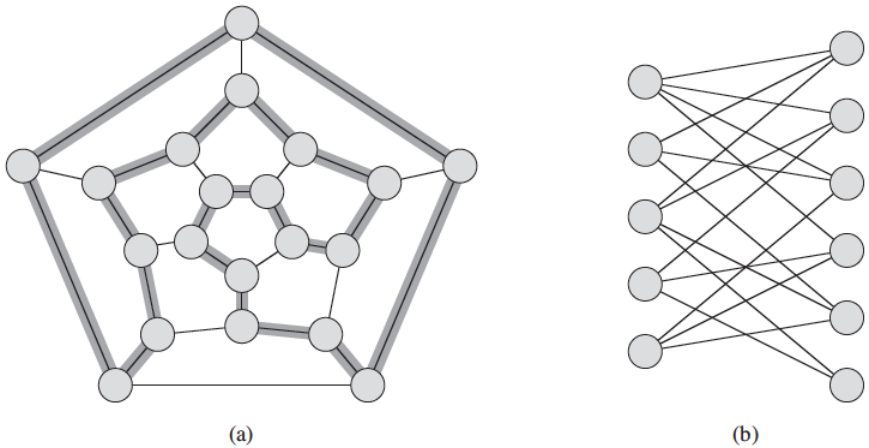

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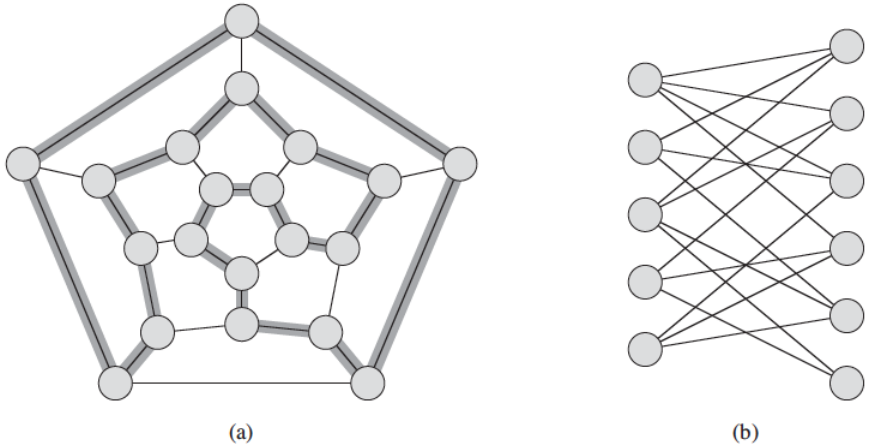

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- Turns out there (probably) aren't any.
- We write HAM-CYCLE(G) as the problem of determining whether or not there is a Hamiltonian cycle in the graph G.

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- Is HAM-CYCLE also in co-NP?
- This is also unknown! Just not having found a cycle isn't a good certificate that there isn't one.

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- Note: There are also problems which aren't in either P or NP – i.e. hard to find a solution, and also hard to verify a solution.

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- If $A(H)$ is "no", then $B(H, e)$ is "yes", since technically every Hamiltonian cycle in H contains e, because there aren't any Hamiltonian cycles :)
- This shows also how if we know the answer to A, we can actually work out *exactly where a Hamiltonian cycle is* since we can one-by-one find edges e that aren't necessary and delete them.

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- (So every NP-complete problem is also NP-hard.)

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- Since we don't know if P = NP, we would like to have a way of saying "This problem is in NP and is really hard unless P = NP."
- This is made more formal by the notion of NP-completeness.
- We say a problem A is *NP-complete* if (a) it is in NP, and (b) every problem B in NP is polynomial-time reducible to A.
- So in some sense A is just as hard as every other problem in NP.
- You may have also heard the term *NP-hard*. That is just condition (b) without condition (a) needing to be true. So a problem that is NP-hard might or might not itself be in NP, but it is possible to reduce any problem in NP to it.
- (So every NP-complete problem is also NP-hard.)
- HAM-CYCLE is an example of a problem that is NP-complete (we won't prove this, but the proof isn't too bad).

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- Since each of those reductions takes polynomial time, the overall reduction must take polynomial time – hence C is polynomial-reducible to B, so B is NP-complete.

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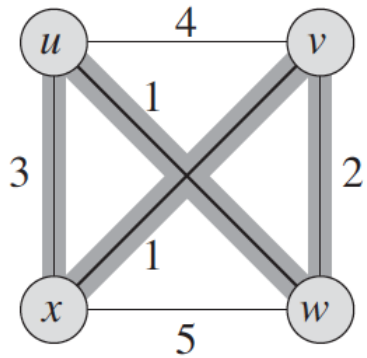

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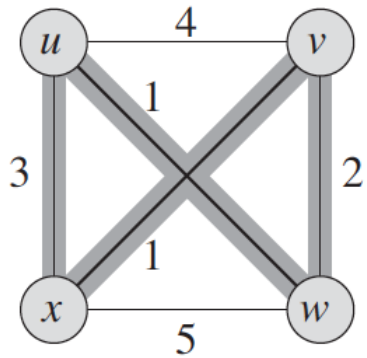

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- Applications of TSP in logistics and transportation, DNA sequencing, even astronomy (efficient ways of moving a telescope to look at different points in the sky).

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- Then, HAM-CYCLE(H) equals TSP(G, ∞), where we have set $k = \infty$.
- This reduction is clearly polynomial-time (the only time is in setting up the weighted graph).